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On a coupled system of equations describing pulse propagation in quadratic media

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Abstract. In the slowly varying envelope approximation we derive the basic equations that describe the propagation of ultrashort pulses in quadratically nonlinear media in which a wave at a fundamental frequency interacts with its second harmonic. In the governing equations we keep linear terms that account for both second- and third-order dispersion and nonlinear terms describing both nonlinear dispersion and self-steepening of the pulse edge. We then perform the Painlevé singularity structure analysis of the most general system of coupled partial differential equations we derived. In a specific case, when third-order dispersion is negligible, by using a Hirota-like method, we found zero- and one-parameter families of bright (fundamental frequency) and dark (second harmonic) solitary waves which travel at a locked group velocity.

1. Introduction

Parametric interactions of intense light beams in materials with quadratic nonlinearities offer a variety of fascinating phenomena such as the formation of two-wave solitons (or, more properly, parametric solitary waves), and the mutual trapping, dragging and steering of optical beams. Both (1+1) solitary waves (that is, one transverse dimension and one propagation dimension) and higher-dimensional solitons exist in bulk crystals and in optical waveguides made of quadratically nonlinear media [1–20]. Temporal solitary waves appear to be more difficult to form with currently available experimental conditions, but both $(1+1)$ and $(2+1)$ spatial solitary waves and the mutual trapping and dragging of two-dimensional spatial solitary waves in a quadratic medium have been recently observed in second harmonic generation experiments [21–23].

In this paper we concentrate on the so-called degenerate case of parametric interactions of optical waves in quadratic nonlinear media in which a wave at a fundamental frequency interacts with its second harmonic. We consider that the dispersive and nonlinear effects are comparable in magnitude, thus the formation of temporal solitons (or, more properly, temporal solitary waves) is possible by the mutual trapping of the fundamental and second harmonic. The parametric solitary waves have potential applications in practical all-optical logic gates because of lower power levels for the soliton formation by using a waveguide made of a quadratic nonlinear medium as compared with a waveguide made of a cubic nonlinear medium.

From Maxwell's equations, by using the usual slowly varying envelope approximation we derive a system of coupled equations governing the propagation of short pulses in a quadratic medium. In the basic equations we keep both a linear term that describes the

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second-order dispersion and the higher-order linear term that accounts for the third-order dispersion. Moreover, we keep nonlinear terms that describe the interaction between the fundamental wave and its second harmonic and the higher-order terms that account for both the nonlinear dispersion and the self-steepening of the pulse edge.

We apply the Painlevé singularity structure analysis to the most general governing equations we derived in order to find out whether the coupled nonlinear partial differential equations pass the Painlevé test for integrability. The Painlevé analysis, as introduced by Weiss, Tabor and Carnevale (WTC) in [24], is one of the systematic methods to identify the integrable cases of the nonlinear partial differential equations [24–30], that is, to check whether the solutions are free from movable critical manifolds. Later, this method was further improved and refined (see, e.g. [31] and the references therein) but, as one will see, it is enough to use the WTC method to prove that the system of partial differential equations which we derived in this paper fails to pass the Painlevé test.

The paper is organized as follows. In section 2 we derive the governing system of partial differential equations describing the evolution of short pulses in a quadratic medium. In section 3 the Painlevé test is performed in detail in the case of the most general system. The results of the Painlevé analysis for other simpler models describing the propagation of optical pulses in quadratic media are also briefly discussed. In section 4 a Hirota-like method [32] is used to find out the exact solitary wave solution in a specific case when third-order dispersion is negligible. In section 5 we briefly present the conclusions.

2. Derivation of the governing equations

In this section we obtain the equations that describe the evolution of short optical pulses in quadratic media. The derivation closely follows the one performed by Menyuk *et al* [6], the difference is that we keep terms in third order in the linear contribution and first order in the nonlinear contribution.

We assume that the fundamental and the second-harmonic waves are propagating in the z-direction and are tightly confined in the transverse direction. Thus, we have a waveguide structure characterized by the normal modes Ψ_1 and Ψ_2 for the fundamental and second-harmonic waves, respectively. The electric fields of the fundamental and the second-harmonic waves are:

$$\begin{aligned} \mathcal{E}_1(x, y, z, t) &= E_1(z, t)\Psi_1(x, y) \\ \mathcal{E}_2(x, y, z, t) &= E_2(z, t)\Psi_2(x, y). \end{aligned} \tag{1}$$

We then write:

$$\begin{aligned} E_1(z, t) &= A_1(z, t) \exp[\mathrm{i}k_1(\omega_0)z - \mathrm{i}\omega_0 t] \\ E_2(z, t) &= A_2(z, t) \exp[\mathrm{i}k_2(2\omega_0)z - 2\mathrm{i}\omega_0 t] \end{aligned} \tag{2}$$

where ω_0 is the central frequency of the fundamental wave, and $k_1(\omega)$ and $k_2(\omega)$ are the wavenumbers of the fundamental and second harmonic, respectively. Here the indices of the wavenumbers stand for the different branches of the dispersion relation. The total electric field is:

$$\boldsymbol{E}(z, t) = E_1(z, t)\boldsymbol{e}_1 + E_1{}^*(z, t)\boldsymbol{e}_1^* + E_2(z, t)\boldsymbol{e}_2 + E_2{}^*(z, t)\boldsymbol{e}_2^* \tag{3}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are complex unit vectors that describe the polarization of the modes at ω_0 and $2\omega_0$. We assume that the electric displacement vector $\boldsymbol{D}$ has contributions that are both linear and quadratic, thus we may set

$$\boldsymbol{D}(z, t) = \boldsymbol{E}(z, t) + 4\pi \int_{-\infty}^{t} \mathrm{d}t' \boldsymbol{\chi}^{(1)}(t - t') \cdot \boldsymbol{E}(\boldsymbol{z}, \boldsymbol{t})$$

$$+4\pi \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t} \mathrm{d}t'' \boldsymbol{\chi}^{(2)}(t-t', t-t'') \cdot \boldsymbol{E}(\boldsymbol{z}, \boldsymbol{t}')\boldsymbol{E}(\boldsymbol{z}, \boldsymbol{t}'') \tag{4}$$

where $\boldsymbol{\chi}^{(1)}$ is a second-rank tensor and $\boldsymbol{\chi}^{(2)}$ is a third-rank tensor. The total electric displacement vector can be written as

$$\boldsymbol{D}(z,t) = D_1(z,t)\boldsymbol{e}_1 + D_1^*(z,t)\boldsymbol{e}_1^* + D_2(z,t)\boldsymbol{e}_2 + D_2^*(z,t)\boldsymbol{e}_2^* \tag{5}$$

where

$$\begin{aligned} D_1(z,t) &= U_1(z,t)\exp[\mathrm{i}k_1(\omega_0)z - \mathrm{i}\omega_0 t] \\ D_2(z,t) &= U_2(z,t)\exp[\mathrm{i}k_2(2\omega_0)z - 2\mathrm{i}\omega_0 t]. \end{aligned} \tag{6}$$

Next we define

$$\begin{aligned} {\chi_1}^{(1)}(\omega) &= \int_{-\infty}^{\infty} \mathrm{d}t\, [\boldsymbol{e}_1{}^* \cdot \boldsymbol{\chi}^{(1)} \cdot \boldsymbol{e}_1] \exp(\mathrm{i}\omega t) \\ {\chi_2}^{(1)}(\omega) &= \int_{-\infty}^{\infty} \mathrm{d}t\, [\boldsymbol{e}_2{}^* \cdot \boldsymbol{\chi}^{(1)} \cdot \boldsymbol{e}_2] \exp(\mathrm{i}\omega t) \end{aligned} \tag{7}$$

with $\boldsymbol{\chi}^{(1)}(t) = 0$ if $t < 0$. We also define

$$\begin{aligned} {\chi_1}^{(2)}(2\omega, -\omega) &= \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t'\, [\boldsymbol{e}_1{}^* \cdot \boldsymbol{\chi}^{(2)}(t,t') \cdot \boldsymbol{e}_1^* \boldsymbol{e}_2] \exp(2\mathrm{i}\omega t' - \mathrm{i}\omega t) \\ {\chi_2}^{(2)}(\omega, \omega) &= \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t' [\boldsymbol{e}_2{}^* \cdot \boldsymbol{\chi}^{(2)}(t,t') \cdot \boldsymbol{e}_1 \boldsymbol{e}_1] \exp(\mathrm{i}\omega t' + \mathrm{i}\omega t). \end{aligned} \tag{8}$$

In the following we proceed as in [6] and we define the following quantities:

$$\begin{aligned} \epsilon_1 &= 1 + 4\pi {\chi_1}^{(1)}(\omega_0) \qquad {\epsilon_1}' = 4\pi \left. \frac{\partial {\chi_1}^{(1)}}{\partial \omega} \right|_{\omega_0} \\ {\epsilon_1}'' &= 4\pi \left. \frac{\partial^2 {\chi_1}^{(1)}}{\partial \omega^2} \right|_{\omega_0} \qquad {\epsilon_1}''' = 4\pi \left. \frac{\partial^3 {\chi_1}^{(1)}}{\partial \omega^3} \right|_{\omega_0} \end{aligned} \tag{9}$$

$$\begin{aligned} \epsilon_2 &= 1 + 4\pi {\chi_2}^{(1)}(\omega_0) \qquad {\epsilon_2}' = 4\pi \left. \frac{\partial {\chi_2}^{(1)}}{\partial \omega} \right|_{2\omega_0} \\ {\epsilon_2}'' &= 4\pi \left. \frac{\partial^2 {\chi_2}^{(1)}}{\partial \omega^2} \right|_{2\omega_0} \qquad {\epsilon_2}''' = 4\pi \left. \frac{\partial^3 {\chi_2}^{(1)}}{\partial \omega^3} \right|_{2\omega_0} \end{aligned} \tag{10}$$

$$\lambda_1 = 4\pi \left. \frac{\partial {\chi_1}^{(2)}}{\partial \omega_1} \right|_{2\omega_0, -\omega_0} \qquad \lambda_2 = 4\pi \left. \frac{\partial {\chi_1}^{(2)}}{\partial \omega_2} \right|_{2\omega_0, -\omega_0}$$

$$\lambda = 4\pi \left. \frac{\partial {\chi_2}^{(2)}}{\partial \omega_1} \right|_{\omega_0, \omega_0} = 4\pi \left. \frac{\partial {\chi_2}^{(2)}}{\partial \omega_2} \right|_{\omega_0, \omega_0} \tag{11}$$

$${\epsilon_1}^{(2)} = 4\pi {\chi_1}^{(2)}(2\omega_0, -\omega_0) \qquad {\epsilon_2}^{(2)} = 4\pi {\chi_2}^{(2)}(\omega_0, \omega_0) \tag{12}$$

$$k_1' = \left. \frac{\mathrm{d}k_1}{\mathrm{d}\omega} \right|_{\omega_0} \qquad k_1'' = \left. \frac{\mathrm{d}^2 k_1}{\mathrm{d}\omega^2} \right|_{\omega_0} \qquad k_1''' = \left. \frac{\mathrm{d}^3 k_1}{\mathrm{d}\omega^3} \right|_{\omega_0} \tag{13}$$

$$k_2' = \left. \frac{\mathrm{d}k_2}{\mathrm{d}\omega} \right|_{2\omega_0} \qquad k_2'' = \left. \frac{\mathrm{d}^2 k_2}{\mathrm{d}\omega^2} \right|_{2\omega_0} \qquad k_2''' = \left. \frac{\mathrm{d}^3 k_2}{\mathrm{d}\omega^3} \right|_{2\omega_0} \tag{14}$$

$$K_1 = \frac{\omega_0^2}{k_1 c^2} \epsilon_1^{(2)} \qquad K_2 = \frac{2\omega_0^2}{k_2 c^2} \epsilon_2^{(2)}. \tag{15}$$

Finally, we obtain the following system of coupled partial differential equations using the slowly varying envelope approximation and keeping terms in third order in the linear contribution and first order in the nonlinear contribution:

$$\mathrm{i}\frac{\partial A_1}{\partial z} + \mathrm{i}k_1'\frac{\partial A_1}{\partial t} - \frac{1}{2}k_1''\frac{\partial^2 A_1}{\partial t^2} - \frac{\mathrm{i}}{6}\left(k_1''' + \frac{3k_1''k_1'}{k_1}\right)\frac{\partial^3 A_1}{\partial t^3}\exp(-\mathrm{i}\Delta kz) + \frac{2\mathrm{i}\omega_0}{k_1c^2}\epsilon_1^{(2)}\frac{\partial}{\partial t}\left(A_1^*A_2\right)\exp(-\mathrm{i}\Delta kz) + \mathrm{i}\frac{\omega_0{}^2}{k_1c^2}\left(\lambda_1 A_1^*\frac{\partial A_2}{\partial t} + \lambda_2 A_2\frac{\partial A_1^*}{\partial t}\right) \times \exp(-\mathrm{i}\Delta kz) + K_1A_1^*A_2\exp(-\mathrm{i}\Delta kz) = 0 \qquad (16)$$

$$\mathrm{i}\frac{\partial A_2}{\partial z} + \mathrm{i}k_2'\frac{\partial A_2}{\partial t} - \frac{1}{2}k_2''\frac{\partial^2 A_2}{\partial t^2} - \frac{\mathrm{i}}{6}\left(k_2''' + \frac{3k_2''k_2'}{k_2}\right)\frac{\partial^3 A_2}{\partial t^3}\exp(\mathrm{i}\Delta kz) + \frac{2\mathrm{i}\omega_0}{k_2c^2}\epsilon_2^{(2)}\frac{\partial}{\partial t}(A_1^2)\exp(\mathrm{i}\Delta kz) + \lambda\frac{2\mathrm{i}\omega_0^2}{k_2c^2}\frac{\partial}{\partial t}(A_1^2)\exp(\mathrm{i}\Delta kz) + K_2A_1^2\exp(\mathrm{i}\Delta kz) = 0 \qquad (17)$$

where $\Delta k = 2k_1(\omega_0) - k_2(2\omega_0)$.

These equations can be put into a normalized form:

$$\mathrm{i}\frac{\partial a_1}{\partial \xi} - \frac{r}{2}\frac{\partial^2 a_1}{\partial s^2} - \frac{\mathrm{i}}{6}\frac{\left(k_1''' + \frac{3k_1''k_1'}{k_1}\right)}{\tau|k_1''|}\frac{\partial^3 a_1}{\partial s^3} + a_1^*a_2\exp(-\mathrm{i}\beta\xi) + \frac{2\mathrm{i}}{\omega_0\tau}\frac{\partial}{\partial s}(a_1^*a_2)\exp(-\mathrm{i}\beta\xi) + \mathrm{i}\left(\frac{\lambda_1}{\epsilon_1^{(2)}\tau}a_1^*\frac{\partial a_2}{\partial s} + \frac{\lambda_2}{\epsilon_1^{(2)}\tau}\frac{\partial a_1^*}{\partial s}a_2\right)\exp(-\mathrm{i}\beta\xi) = 0 \qquad (18)$$

$$\mathrm{i}\frac{\partial a_2}{\partial \xi} - \mathrm{i}\delta\frac{\partial a_2}{\partial s} - \frac{\alpha}{2}\frac{\partial^2 a_2}{\partial s^2} - \frac{\mathrm{i}}{6}\frac{\left(k_2''' + \frac{3k_2''k_2'}{k_2}\right)}{\tau|k_1''|}\frac{\partial^3 a_2}{\partial s^3} + a_1^2\exp(\mathrm{i}\beta\xi) + \frac{\mathrm{i}}{\omega_0\tau}\frac{\partial}{\partial s}(a_1^2)\exp(\mathrm{i}\beta\xi) + \mathrm{i}\frac{\lambda}{\epsilon_2^{(2)}\tau}\frac{\partial}{\partial s}(a_1^2)\exp(\mathrm{i}\beta\xi) = 0 \qquad (19)$$

where

$$\xi = \frac{|k_1''|}{\tau^2}z \qquad s = \frac{t}{\tau} - \frac{k_1'}{\tau}z \qquad \delta = \frac{(k_1' - k_2')\tau}{|k_1''|} \qquad \alpha = \frac{k_2''}{|k_1''|}$$
$$\beta = \frac{\Delta k\tau^2}{|k_1''|} \qquad a_1 = \frac{|K_1K_2|^{1/2}\tau^2}{|k_1''|}A_1 \qquad a_2 = \frac{K_1\tau^2}{|k_1''|}A_2 \qquad r = \mathrm{sgn}(k_1''). \qquad (20)$$

We mention that the last three terms in equation (18) account for the self-steepening of the pulse edge and the nonlinear dispersion. Analogously, the last two terms in equation (19) account for the self-steepening of the pulse edge and the nonlinear dispersion, respectively. In equations (18)–(20), $r = -1$ for the case of anomalous dispersion at the fundamental frequency and $r = 1$ for the case of normal dispersion at the fundamental frequency, α is the ratio of the second-order dispersions at the two frequencies, δ is the group-velocity mismatch parameter, τ is the input pulse duration, and β is the wavevector mismatch parameter.

In order to observe these temporal solitary waves one must first have materials with strong quadratic nonlinearities, so that the waves undergo a 2π phase shift after propagating a few millimetres in a waveguide of the order of 1 cm. In order to have solitary wave formation we need a sufficiently large second-order dispersion to compensate for the quadratic nonlinearity. As an example, we consider a $LiNbO_3$ waveguide for which $\chi^{(2)} = 12$ pm V^{-1} and the wavelength of the fundamental wave is $\lambda = 1.06$ μm. The typical second-order dispersion is $|k_1''| = 0.1$ ps^2 m^{-1}. For an input pulse duration $\tau = 15$ fs we have a dispersion length $l_d = \frac{\tau^2}{|k_1''|} \simeq 2$ mm, much less than the typical length of a few centimetres of the sample. Thus, for input beam profiles deviating strongly from the solitary wave shape, a sample of a few centimetres long allows the formation of parametric solitary waves. At the same time, with such an ultrashort pulse the higher-order terms in equations (18), (19), that is, the terms accounting for the self-steepening of the pulse edge and the nonlinear dispersion come into the play.

We mention that in another physical setting, coupled partial differential equations which extend the nonlinear Schrödinger equation and describe the dynamics of femtosecond optical pulses in birefringent fibres were introduced [33, 34] and although, in general, they do not pass the Painlevé test for integrability, coupled solitary waves were provided by using the Hirota technique [34].

3. Painlevé singularity structure analysis

In order to check for the integrability of a system of partial differential equations we analyse whether the system has the Painlevé property as introduced by Weiss *et al* [24]. The method involves expanding the solution in a Laurent series about a singular or pole manifold. Also, the method gives rise to a powerful formalism from which one may deduce the Lax pairs, the Bäcklund transformations, the Hirota equations, the motion invariants, symmetries and commuting flows, and the geometrical structure of the phase space [30].

Next we rewrite the system (18), (19) in the form:

$$\begin{aligned} &\mathrm{i}q_{1x} - \frac{r}{2}q_{1tt} + q_1^* q_2 \mathrm{e}^{-\mathrm{i}\beta x} + \mathrm{i}\gamma_1 q_{1ttt} + \mathrm{i}\alpha_1 q_{1t}^* q_2 \mathrm{e}^{-\mathrm{i}\beta x} + \mathrm{i}\alpha_2 q_1^* q_{2t} \mathrm{e}^{-\mathrm{i}\beta x} = 0 \\ &\mathrm{i}q_{2x} - \mathrm{i}\delta q_{2t} - \frac{\alpha}{2} q_{2tt} + \mathrm{i}\gamma_2 q_{2ttt} + q_1^2 \mathrm{e}^{\mathrm{i}\beta x} + \mathrm{i}\gamma_3 (q_1^2)_t \mathrm{e}^{\mathrm{i}\beta x} = 0 \end{aligned} \tag{21}$$

where

$$\gamma_{1,2} = -\frac{1}{6}\frac{\left(k_{1,2}''' + \frac{3k_{1,2}''k_{1,2}'}{k_{1,2}}\right)}{\tau|k_1'''|} \qquad \alpha_{1,2} = \frac{2}{\omega_0\tau} + \frac{\lambda_{1,2}}{\epsilon_1^{(2)}\tau} \qquad \gamma_3 = \frac{1}{\omega_0\tau} + \frac{\lambda}{\epsilon_2^{(2)}\tau}. \tag{22}$$

Here the subscripts x and t denote the partial derivatives with respect to x and t.

In this section we perform the Painlevé test for the following system of coupled partial differential equations:

$$\begin{aligned} &\mathrm{i}q_{1x} - \frac{r}{2}q_{1tt} + q_1^* q_2 + \mathrm{i}\gamma_1 q_{1ttt} + \mathrm{i}\alpha_1 q_1^* q_{2t} + \mathrm{i}\alpha_2 q_{1t}^* q_2 = 0 \\ &\mathrm{i}q_{2x} - \mathrm{i}\delta q_{2t} - \frac{\alpha}{2} q_{2tt} + q_1^2 + \mathrm{i}\gamma_2 q_{2ttt} + \mathrm{i}\gamma_3 (q_1^2)_t = 0. \end{aligned} \tag{23}$$

For the sake of simplicity we took the normalized wavevector mismatch parameter $\beta = 0$ in system (21) without restricting the generality of the Painlevé singularity structure analysis.

In order to study the integrability properties of system (23), we rewrite it in terms of four complex functions a, b, c, and d defined by $q_1 = a$, $q_2 = b$, $q_1^* = c$, $q_2^* = d$. Consequently, we have the following equations:

$$
\begin{aligned}
&\mathrm{i}a_x - \frac{r}{2}a_{tt} + cb + \mathrm{i}\gamma_1 a_{ttt} + \mathrm{i}\alpha_1 cb_t + \mathrm{i}\alpha_2 c_t b = 0 \\
&\mathrm{i}b_x - \mathrm{i}\delta b_t - \frac{\alpha}{2}b_{tt} + a^2 + \mathrm{i}\gamma_2 b_{ttt} + \mathrm{i}\gamma_3 (a^2)_t = 0 \\
&-\mathrm{i}c_x - \frac{r}{2}c_{tt} + ad - \mathrm{i}\gamma_1 c_{ttt} - \mathrm{i}\alpha_1 ad_t - \mathrm{i}\alpha_2 a_t d = 0 \\
&-\mathrm{i}d_x - \frac{\alpha}{2}d_{tt} + \mathrm{i}\delta d_t + c^2 - \mathrm{i}\gamma_2 d_{ttt} - \mathrm{i}\gamma_3 (c^2)_t = 0.
\end{aligned}
\tag{24}
$$

The Painlevé analysis in the formulation of Weiss *et al* [24] essentially consists of three stages: (i) determining the leading-order behaviour; (ii) identifying the resonances, and (iii) verifying that a sufficient number of arbitrary functions exists without the introduction of movable critical singularity manifolds. To start with, let us introduce the following series for the four functions a, b, c, d:

$$
\begin{aligned}
&a = \Phi^{p_1}\sum_{j\geqslant 0} a_j(x)\Phi^j \qquad b = \Phi^{p_2}\sum_{j\geqslant 0} b_j(x)\Phi^j \\
&c = \Phi^{p_3}\sum_{j\geqslant 0} c_j(x)\Phi^j \qquad d = \Phi^{p_4}\sum_{j\geqslant 0} d_j(x)\Phi^j
\end{aligned}
\tag{25}
$$

with the Kruskal ansatz [25]: $\Phi(x,t) = t - \Psi(x)$. Here $p_i, 1 \leqslant i \leqslant 4$ are negative integers. By introducing the above series into equations (23) and equating the leading terms we obtain:

$$
\begin{aligned}
&p_1 - 3 = p_2 + p_3 - 1 \\
&p_2 - 3 = 2p_1 - 1 \\
&p_3 - 3 = p_1 + p_4 - 1 \\
&p_4 - 3 = 2p_3 - 1.
\end{aligned}
\tag{26}
$$

From this system we obtain the unique solution $p_i = -2, 1 \leqslant i \leqslant 4$ and the following equations for the functions a_0, b_0, c_0 and d_0:

$$
\begin{aligned}
&12\gamma_1 a_0 + b_0 c_0(\alpha_1 + \alpha_2) = 0 \\
&6\gamma_2 b_0 + \gamma_3 a_0^2 = 0 \\
&12\gamma_1 c_0 + a_0 d_0(\alpha_1 + \alpha_2) = 0 \\
&6\gamma_2 d_0 + \gamma_3 c_0^2 = 0.
\end{aligned}
\tag{27}
$$

In system (27) not all of the four functions are independent. This can easily be seen, as independent equations can be chosen for the first two equations of system (27) and the equation: $b_0 c_0^2 - a_0^2 d_0 = 0$. Therefore, at this stage we conclude that one of these four functions is arbitrary. Now we find the resonances, that is, the powers at which the arbitrary functions can enter into the series. Thus we substitute the following expressions into the coupled system (23):

$$
\begin{aligned}
&a = a_0\Phi^{-2} + a_j\Phi^{j-2} \\
&b = b_0\Phi^{-2} + b_j\Phi^{j-2} \\
&c = a_0\Phi^{-2} + c_j\Phi^{j-2} \\
&d = a_0\Phi^{-2} + d_j\Phi^{j-2}.
\end{aligned}
\tag{28}
$$

Keeping the leading-order terms together, we obtain a linear system of four algebraic equations in a_j, b_j, c_j, and d_j:

$$\begin{aligned}
&\gamma_1(j-2)(j-3)(j-4)a_j + [\alpha_1(j-2) - 2\alpha_2]c_0b_j + [\alpha_2(j-2) - 2\alpha_1]b_0c_j = 0 \\
&\gamma_1(j-2)(j-3)(j-4)c_j + [\alpha_1(j-2) - 2\alpha_2]d_0a_j + [\alpha_2(j-2) - 2\alpha_1]a_0d_j = 0 \\
&\gamma_2(j-2)(j-3)(j-4)b_j + 2\gamma_3(j-4)a_0a_j = 0 \\
&\gamma_2(j-2)(j-3)(j-4)d_j + 2\gamma_3(j-4)c_0c_j = 0.
\end{aligned} \tag{29}$$

To have a non-trivial solution for a_j, b_j, c_j, and d_j, the corresponding determinant must be zero:

$$\begin{aligned}
\Delta(j,r) = {} & (1+r)^2(j-2)^4(j-3)^4(j-4)^2 \\
& -288[j-2(1+r)](1+r)(j-2)^2(j-3)^2(j-4) \\
& +144^2[j-2(1+r)]^2 - 144[rj-2(1+r)]^2(j-2)^2(j-3)^2 = 0
\end{aligned} \tag{30}$$

where $r = \alpha_1/\alpha_2$. It is easily verified that $\Delta(j,r) = P_1(j,r)P_2(j,r)$, where

$$P_1(j,r) = (j+1)(j-6)[j^3(1+r) - 9j^2(1+r) + j(26r+38) - 48(r+1)] \tag{31}$$

$$P_2(j,r) = j(j^2 - 5j + 18)[j^2(1+r) - 9j(1+r) + 26r + 14]. \tag{32}$$

The resonance at $j = -1$ corresponds to the arbitrariness of Φ itself and the resonance at $j = 0$ corresponds to the fact that one of the four functions a_0, b_0, c_0, and d_0 is arbitrary. Because at least two of the zeros of the polynomial $P_2(j,r)$ are complex, we conclude that the coupled system (23) fails to pass the Painlevé test.

Next we briefly present the Painlevé analysis for other two coupled nonlinear partial differential equations coming from system (23). First, we perform the Painlevé singularity analysis for the following equations [6–10]:

$$\begin{aligned}
&\mathrm{i}q_{1x} - \frac{r}{2}q_{1tt} + q_1^*q_2 = 0 \\
&\mathrm{i}q_{2x} - \mathrm{i}\delta q_{2t} - \frac{\alpha}{2}q_{2tt} + q_1^2 = 0.
\end{aligned} \tag{33}$$

These coupled equations describe the self-action of light in parametric wave interactions in nonlinear quadratic media in the presence of different group velocities between the fundamental frequency wave and the second harmonic when the third-order dispersion, as well as the nonlinear terms which account for the self-steepening of the pulse edge and the nonlinear dispersion, are negligible.

In this case, in a similar way as before, we found that the equation for the resonances is as follows

$$(j+1)j(j-5)(j-6)(j^2 - 5j + 12)(j^2 - 5j + 18) = 0. \tag{34}$$

Because four of the resonances are complex we conclude that this system does not possess the Painlevé property.

Although system (33) fails to pass the Painlevé test, the existence has recently [20] been shown of a two-parameter family of 'bright–bright' solitary waves which travel at a locked velocity. These parametric solitary waves are chirped and form due to the interplay of the second-order dispersion, the nonlinear interaction and the temporal walk-off which comes from the different group velocities of the interacting waves.

Finally, we perform the Painlevé test for the following system:

$$\begin{aligned}
&\mathrm{i}q_{1x} - \frac{r}{2}q_{1tt} + q_1^*q_2 + \mathrm{i}\alpha_1 q_1^* q_{2t} + \mathrm{i}\alpha_2 q_{1t}^* q_2 = 0 \\
&\mathrm{i}q_{2x} - \mathrm{i}\delta q_{2t} - \frac{\alpha}{2}q_{2tt} + q_1^2 + \mathrm{i}\gamma_3(q_1^2)_t = 0.
\end{aligned} \tag{35}$$

This system comes from the general system (23) by neglecting the terms accounting for the third-order dispersion. Imposing the resonances to be integers we obtain $\alpha_1 = \alpha_2$. Thus, in this case the resonances are found to be: $j = -1, 0, 2, 2, 2, 2, 2, 3$. However, the system does not pass the Painlevé test because for the resonance $j = 0$ all the four functions a_0, b_0, c_0 and d_0 are fixed instead of one of them being arbitrary. Although system (35) does not pass the Painlevé test, we will be able to find a symbiotic pair of bright (fundamental frequency) and dark (second harmonic) solitary waves by using the direct Hirota-like method.

4. Hirota-like method and the exact 'bright–dark' solitary wave solutions

In this section we shall find out the exact solitary wave solutions of the following system of coupled equations by using a Hirota-like direct method [32]:

$$\begin{aligned} &\mathrm{i}q_{1x} - \frac{r}{2}q_{1tt} + q_1^* q_2 \mathrm{e}^{-\mathrm{i}\beta x} + \mathrm{i}d(q_1^* q_2)_t \mathrm{e}^{-\mathrm{i}\beta x} = 0 \\ &\mathrm{i}q_{2x} - \mathrm{i}\delta q_{2t} - \frac{\alpha}{2}q_{2tt} + q_1^2 \mathrm{e}^{\mathrm{i}\beta x} + \mathrm{i}\gamma_3 (q_1^2)_t \mathrm{e}^{\mathrm{i}\beta x} = 0. \end{aligned} \tag{36}$$

This system comes from the general equations (21) by neglecting the third-order dispersion terms and taking $\alpha_1 = \alpha_2 = d$.

Recently [19], a family of 'bright–dark' parametric solitary waves has been found in a second-harmonic generation setting. These solitary waves form from the interplay between the group-velocity difference and the nonlinear interaction between the two waves (the second-order dispersion at both frequencies being neglected). In the following we find that the interplay of the second-order dispersion, group-velocity difference, quadratic nonlinearity and self-steepening effect does permit us to be sustained symbiotic solitary waves that are of the bright type for the fundamental frequency wave and of the dark type for the second harmonic.

In order to construct the Hirota-like bilinear form we first introduce the new functions $a_{1,2}(x, t)$:

$$\begin{aligned} q_1(x, t) &= a_1(x, t) \exp[\mathrm{i}(\omega_1 t + \kappa_1 x)] \\ q_2(x, t) &= a_2(x, t) \exp[\mathrm{i}(\omega_2 t + \kappa_2 x)] \end{aligned} \tag{37}$$

where $\omega_{1,2}$ are the corrections to the corresponding frequencies and $\kappa_{1,2}$ are the corrections to the corresponding wavenumbers. System (36) becomes:

$$\begin{aligned} &a_{1x} + \nu_1 a_{1t} + \mu_1 a_1 + \mathrm{i}\frac{r}{2}a_{1tt} - \mathrm{i}\gamma_1 a_1^* a_2 \exp[\mathrm{i}(\Omega t + \kappa x)] + d\{a_1^* a_2 \exp[\mathrm{i}(\Omega t + \kappa x)]\}_t = 0 \\ &a_{2x} + \nu_2 a_{2t} + \mu_2 a_2 + \mathrm{i}\frac{\alpha}{2}a_{2tt} - \mathrm{i}\gamma_2 a_1^2 \exp[-\mathrm{i}(\Omega t + \kappa x)] + \gamma_3\{a_1^2 \exp[-\mathrm{i}(\Omega t + \kappa x)]\}_t = 0 \end{aligned} \tag{38}$$

where

$$\begin{aligned} &\nu_1 = -r\omega_1 \qquad \mu_1 = \mathrm{i}\left(\kappa_1 - \frac{r\omega_1^2}{2}\right) \qquad \gamma_1 = 1 - d\omega_1 \\ &\nu_2 = -(\delta + \alpha\omega_2) \qquad \mu_2 = \mathrm{i}\left(\kappa_2 - \delta\omega_2 - \frac{\alpha\omega_2^2}{2}\right) \qquad \gamma_2 = 1 - \gamma_3\omega_2 \\ &\Omega = \omega_2 - 2\omega_1 \qquad \kappa = \kappa_2 - 2\kappa_1 - \beta. \end{aligned} \tag{39}$$

Let us now take the Hirota transformation in the form:

$$a_1(x, t) = \frac{F(x, t)}{G(x, t)} \qquad a_2(x, t) = \frac{E(x, t)}{G(x, t)} \tag{40}$$

where F, E are complex functions, and G is a real function. Using the Hirota bilinear operators:

$$D_t^m D_x^n(FG) \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) F(x,t)G(x',t')|_{t'=t,x'=x} \tag{41}$$

system (38) becomes:

$$\begin{aligned}
&\frac{1}{G^2}(D_x + \nu_1 D_t + \mu_1)(FG) - \mathrm{i}\gamma_1 \frac{F^* E}{G^2} \exp[\mathrm{i}(\Omega t + \kappa x)] \\
&\qquad + \frac{\partial}{\partial t}\left(\mathrm{i}\frac{r}{2}\frac{1}{G^2} D_t(FG) + d\frac{F^* E}{G^2} \exp[\mathrm{i}(\Omega t + \kappa x)]\right) = 0 \\
&\frac{1}{G^2}(D_x + \nu_2 D_t + \mu_2)(EG) - \mathrm{i}\gamma_2 \frac{F^2}{G^2} \exp[-\mathrm{i}(\Omega t + \kappa x)] \\
&\qquad + \frac{\partial}{\partial t}\left(\mathrm{i}\frac{\alpha}{2}\frac{1}{G^2} D_t(EG) + \gamma_3\frac{F^2}{G^2} \exp[-\mathrm{i}(\Omega t + \kappa x)]\right) = 0.
\end{aligned} \tag{42}$$

We see that these equations can be satisfied if we impose:

$$\begin{aligned}
&\mathrm{i}\frac{r}{2} D_t(FG) + dF^* E \exp[\mathrm{i}(\Omega t + \kappa x)] = 0 \\
&\mathrm{i}\frac{\alpha}{2} D_t(EG) + \gamma_3 F^2 \exp[-\mathrm{i}(\Omega t + \kappa x)] = 0 \\
&(D_x + \nu_1 D_t + \mu_1)(FG) - \mathrm{i}\gamma_1 F^* E \exp[\mathrm{i}(\Omega t + \kappa x)] = 0 \\
&(D_x + \nu_2 D_t + \mu_2)(EG) - \mathrm{i}\gamma_2 F^2 \exp[-\mathrm{i}(\Omega t + \kappa x)] = 0.
\end{aligned} \tag{43}$$

From (43) we easily obtain:

$$\begin{aligned}
&(D_x + \rho_1 D_t + \mu_1)(FG) = 0 \\
&(D_x + \rho_2 D_t + \mu_2)(EG) = 0
\end{aligned} \tag{44}$$

where $\rho_1 = \nu_1 - \frac{\gamma_1 r}{2d}$ and $\rho_2 = \nu_2 - \frac{\gamma_2 \alpha}{2\gamma_3}$. In order to obtain the exact solitary wave solutions, we proceed in the standard way. Thus, we assume:

$$F = \varepsilon f_1 \qquad G = 1 + \varepsilon^2 g_1 \qquad E = e_0 + \varepsilon^2 e_1 \tag{45}$$

where ε is an arbitrary parameter which is finally set equal to 1. Here e_0 is a complex constant, g_1 is a real function and f_1 and e_1 are complex functions. Substituting (45) into system (43) and then collecting the terms with the same power in ε we are left with a system of coupled equations for f_1, g_1, e_0 and e_1. We seek for the functions g_1, f_1 and e_1 having an exponential dependence of the variables x and t. From the compatibility conditions for the system of coupled equations for f_1, g_1, e_1 and e_0 it results that:

$$\begin{aligned}
&\kappa_2 - \delta\omega_2 - \frac{\alpha\omega_2^2}{2} = 0 \\
&\nu_1 - \frac{\gamma_1 r}{2d} = \nu_2 - \frac{\gamma_2 \alpha}{2\gamma_3} \\
&\mathrm{i}(\kappa + \rho_1 \Omega) + 2\mu_1 = 0.
\end{aligned} \tag{46}$$

One can easily find the expressions for g_1, f_1, e_1 and e_0:

$$g_1(x,t) = \exp(2\xi) \tag{47}$$

$$f_1 = \frac{\eta}{\rho_1}\left(\frac{\alpha r}{d\gamma_3}\right)^{1/2} \exp(\xi) \exp\left[\frac{\mathrm{i}}{2}(\Omega t + \kappa x + \theta)\right] \tag{48}$$

$$e_1 = \left(-\frac{r}{2d}\right)\left(\frac{\eta}{\rho_1} + \mathrm{i}\frac{\Omega}{2}\right)\exp(2\xi)\exp\left[\mathrm{i}\left(\frac{\pi}{2}+\theta\right)\right] \tag{49}$$

$$e_0 = \left(-\frac{r}{2d}\right)\left(-\frac{\eta}{\rho_1} + \mathrm{i}\frac{\Omega}{2}\right)\exp\left[\mathrm{i}\left(\frac{\pi}{2}+\theta\right)\right]. \tag{50}$$

By using these expressions one obtains:

$$a_1(x,t) = \frac{\eta}{2\rho_1}\left(\frac{\alpha r}{d\gamma_3}\right)^{1/2}\mathrm{sech}\,(\xi)\exp\left[\frac{\mathrm{i}}{2}(\Omega t + \kappa x + \theta)\right] \tag{51}$$

$$a_2(x,t) = \left(-\frac{r}{2d}\right)\left[\left(\frac{\eta}{\rho_1}\right)^2 + \frac{\Omega^2}{4}\right]^{1/2}\frac{\sinh(\xi+\mathrm{i}\varphi)}{\cosh(\xi)}\exp\left[\mathrm{i}\left(\frac{\pi}{2}+\theta\right)\right] \tag{52}$$

where

$$\xi = \eta\left(x - \frac{t}{\rho_1}\right) + \xi_0 \qquad \varphi = \arctan\left(\frac{\Omega\rho_1}{2\eta}\right) \qquad \rho_1 = -\frac{r\omega_1}{2} - \frac{r}{2d}$$

with $\Omega = \omega_2 - 2\omega_1$, η, φ, θ being integration constants. Finally, we obtain the following expressions for $q_{1,2}$:

$$q_1(x,t) = \frac{\eta}{2\rho_1}\left(\frac{\alpha r}{d\gamma_3}\right)^{1/2}\mathrm{sech}(\xi)\exp\left[\frac{\mathrm{i}}{2}(\omega_2 t + \kappa_2 x - \beta x + \theta)\right] \tag{53}$$

$$q_2(x,t) = \left(-\frac{r}{2d}\right)\left[\left(\frac{\eta}{\rho_1}\right)^2 + \frac{\Omega^2}{4}\right]^{1/2}\frac{\sinh(\xi+\mathrm{i}\varphi)}{\cosh(\xi)}\exp\left[\mathrm{i}\left(\omega_2 t + \kappa_2 x + \frac{\pi}{2}+\theta\right)\right]. \tag{54}$$

From constraints (46) we obtain the following system of equations for $\omega_{1,2}$:

$$\frac{r\omega_1}{d} - \frac{\alpha\omega_2}{2\gamma_3} = \beta \tag{55}$$

$$\frac{r\omega_1}{2} - \frac{\alpha\omega_2}{2} = \delta - \frac{r}{2d} + \frac{\alpha}{2\gamma_3}. \tag{56}$$

Here we have two distinct cases to analyse:

(a) If $d = 2\gamma_3$ then $\omega_1 = \frac{2\gamma_3}{r}\beta + \frac{\alpha}{r}\omega_2$, ω_2 being a free parameter. In addition, we have a compatibility condition for the linear system (55), (56): $\frac{d\beta}{2} = \delta + \frac{2\alpha - r}{2d}$. In this case the solution (53), (54) constitutes a one-parameter family of bright–dark solitary waves.

(b) If $d \neq 2\gamma_3$ then $\omega_{1,2}$ can be easily obtained by solving the linear system (55), (56). Thus, in this case the corrections $\omega_{1,2}$ to the frequencies of the fundamental wave and the second harmonic are fixed and the solution (53), (54) has no free parameters.

The modulational instability of the background continuous wave is the preferential mechanism of instability of coupled solitary waves which contain dark components. We mention that the symbiotic 'bright–dark' pair of solitary waves found recently in [19] can be modulationally stable. We expect that the 'bright–dark' solitary wave we found can also be modulationally stable. However, the rigorous stability analysis of the 'bright–dark' solitary waves we found is an important issue and deserves a separate study.

In figures 1 and 2 we show the shape of the pair of 'bright–dark' solitary waves for the following set of parameters: $r = -1$, $d = -1$, $\alpha = 1$, $\gamma_3 = 1$, $\frac{\eta}{\rho_1} = 1$, $\delta = 0$. For this choice of the parameters the corrections to the frequencies are fixed and there are no free parameters of the exact solitary wave solution. In figure 1 we show the variation of the intensity as a function of time for the value of the normalized wavevector mismatch

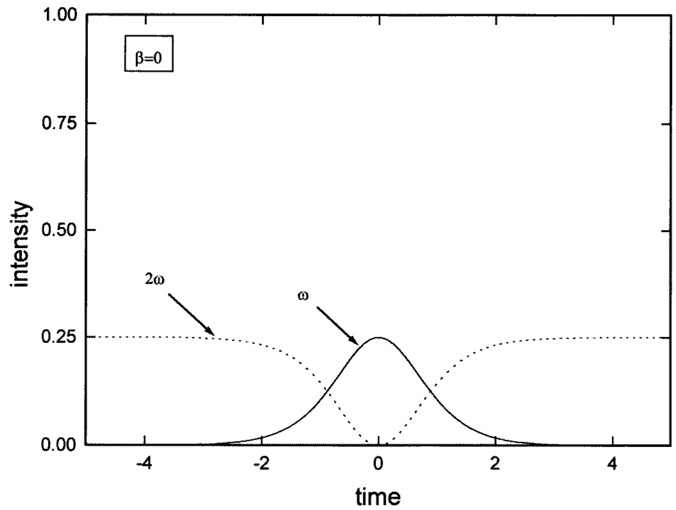

Figure 1. The shape of the intensity of the fundamental wave (full curve) and the second harmonic (dotted curve) for the following set of parameters: $r = -1$, $d = -1$, $\alpha = 1$, $\gamma_3 = 1$, $\frac{\eta}{\rho_1} = 1$, $\delta = 0$. Here the normalized wavevector mismatch parameter $\beta = 0$.

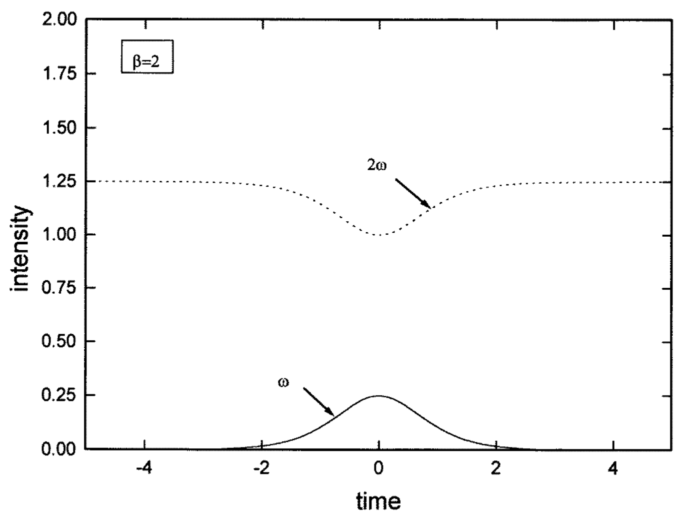

Figure 2. Same as in figure 1 but with the normalized wavevector mismatch parameter $\beta = 2$.

parameter $\beta = 0$. In this case the intensities of the fundamental wave and the second harmonic have the simple expressions:

$$|q_1|^2 = \frac{1}{4}\,\mathrm{sech}^2(t) \qquad |q_2|^2 = \frac{1}{4}\,\tanh^2(t). \tag{57}$$

Figure 2 presents the shape of the intensities of both waves for $\beta = 2$. In this case the fundamental wave and second harmonic have the following intensities:

$$|q_1|^2 = \tfrac{1}{4}\,\mathrm{sech}^2(t)$$
$$|q_2|^2 = \tfrac{1}{4}[\tanh^2(t) + 4]. \tag{58}$$

We see that for this choice of parameters the background intensity for the dark wave increases with the mismatch parameter β.

5. Conclusions

In this paper we have derived the coupled system of partial differential equations describing pulse propagation of ultrashort pulses in a quadratic medium, by keeping linear terms that account for both second- and third-order dispersion and nonlinear terms describing both nonlinear dispersion and self-steepening of the pulse edge. When the third-order dispersion is negligible, by using a Hirota-like method, we have found zero- and one-parameter families of symbiotic solitary waves. This parametric solitary wave has the interesting property that it is of a bright-type concerning the fundamental wave and of a dark-type concerning the second harmonic.

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